Time-varying filter interpretation of Fourier transform and its variants

Prasanta Kumar Ghosh, T.V. Sreenivas*

Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore-560 012, India

Received 21 February 2005; received in revised form 28 September 2005; accepted 16 January 2006
Available online 13 February 2006

Abstract

We show that a stationary signal concept such as the Fourier transform (FT) can be linked to a time-varying (TV) concept such as linear TV filtering. In particular, we give a continuous-time and discrete-time linear TV filter interpretation to continuous-time FT and discrete Fourier transform (DFT), respectively. As a consequence, important variants such as warped DFT, DCT, DST also can be viewed as linear TV filtering operations. Extending these results, we show that short-time Fourier transform (STFT) can be viewed as a linear TV filterbank operation, thereby interpreting time-varying spectrum of a signal as the output of a linear TV filterbank.

© 2006 Elsevier B.V. All rights reserved.

Keywords: Time-varying filter; Fourier transform; Short-time Fourier transform

1. Introduction

Fourier transform (FT) is an important tool in signal processing. Discrete Fourier transform (DFT) leads to an efficient way of computing the Fourier transform of finite duration discrete-time signals and forms the foundation for both analysis and design of discrete-time signals and systems. Though FT in general has no filtering interpretation, DFT has a useful filterbank interpretation [1] due to its inherent matrix operation. Warped Fourier transform (WFT) and warped discrete Fourier transform (WDT) are also useful tools for signal analysis [2,3], although the inverse WFT and inverse WDT cannot be defined easily unlike inverse FT (IFT) and inverse DFT (IDFT). In this paper, we have put all these transforms in a common filtering framework. We have been able to show that these filters are either continuous-time or discrete-time linear time-varying (TV) filters depending on the type of transform.

For analyzing signals and systems with TV properties, short-time Fourier transform (STFT) is widely used as a joint time–frequency representation of the signal. The STFT also has a modulated filterbank interpretation [4]. We interpret STFT as a linear TV filterbank operation and therefore, the TV spectrum obtained by STFT can also be interpreted as the output of a linear TV filterbank.

The paper is organized as follows: Section 2 discusses the continuous-time linear time-varying (CT_LTV) filter interpretation of continuous-time transforms and in Section 3 these are extended to the discrete-time case.
2. Continuous-time case

The FT of a finite energy continuous-time signal \( x(t) \) is given by

\[
X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt.
\] (1)

The signal \( x(t) \) can be obtained from \( X(\Omega) \) through the IFT:

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega.
\] (2)

The WFT has been defined by choosing a warping function \( f(\Omega) \) in place of \( \Omega \) in (1); i.e., WFT of \( x(t) \) is [5]

\[
X(f(\Omega)) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega f(t)} dt.
\] (3)

Unlike the IFT, the inverse WFT (IWFT) cannot be defined easily; however, even in the absence of IWFT, WFT is a useful signal processing tool.

Further, to analyze signal with TV properties, the STFT of a signal \( x(t) \) is defined as

\[
X(t, \Omega_1) = \int_{-\infty}^{\infty} x(\tau) w(t - \tau) e^{-j\Omega_1 \tau} d\tau,
\] (4)

where \( w(t) \) is a localizing window function. The above relation has been interpreted as a modulated filtering operation, i.e., for a specific \( \Omega = \Omega_1 \) (see Fig. 1)

\[
X(t, \Omega_1) = \int_{-\infty}^{\infty} x(\tau) w(t - \tau) e^{-j\Omega_1 \tau} d\tau
= \{ x(t) e^{-j\Omega_1 t} \} \ast w(t).
\] (5)

Also,

\[
X(t, \Omega_1) = \int_{-\infty}^{\infty} x(\tau) w(t - \tau) e^{-j\Omega_1 \tau} d\tau
= e^{-j\Omega_1 t} \int_{-\infty}^{\infty} x(\tau) w(t - \tau) e^{-j\Omega_1 (t - \tau)} d\tau
= e^{-j\Omega_1 t} \{ x(t) \ast w(t) e^{j\Omega_1 t} \}.
\] (6)

We can also define a short-time warped FT (ST_WFT) similar to the STFT as follows:

\[
X(t, f(\Omega)) = \int_{-\infty}^{\infty} x(\tau) w(t - \tau) e^{-j\Omega f(t)} d\tau.
\] (7)

However, unlike STFT, inversion of ST_WFT would be more complicated.

Another tool for synthesizing and analyzing signals with TV properties is that of CT_LTV filter (see Fig. 2). The output \( y(t) \) of a CT_LTV is defined as [6]

\[
y(t) = \int_{-\infty}^{\infty} h(t, t - \tau) x(\tau) d\tau = \int_{-\infty}^{\infty} h(t, \tau) x(\tau - t) d\tau,
\] (8)

where \( x(\tau) \) is the input signal and \( h(t, \tau) \) is the TV impulse response of the filter, whose TV transfer function is \( H(t, \Omega) = \int_{-\infty}^{\infty} h(t, \tau) e^{-j\Omega \tau} d\tau \).

Based on the above concepts, we develop the following propositions which link the stationary concepts of FT to that of TV filtering. The LTV filtering is performed through convolution operation and not through conventional Fourier integral.

**Proposition 1** (The FT can be interpreted as the output of a CT_LTV system). Consider a CT_LTV system with impulse response \( h_{\text{FT}}(t, \tau) = e^{-j\Omega(t-\tau)} \). If the input to this system is the signal \( x(t) \), the output signal of the system is the Fourier transform of \( x(t) \), i.e., \( y(t) = X(t) \) or \( y(\Omega) = X(\Omega) \).

**Proof.**

\[
y(t) = \int_{-\infty}^{\infty} h_{\text{FT}}(t, t - \tau) x(\tau) d\tau
= \int_{-\infty}^{\infty} e^{-j\Omega_1 \tau} x(\tau) d\tau
= \int_{-\infty}^{\infty} x(\tau) e^{-j\Omega_1 \tau} d\tau
= X(t) \quad \text{(by definition)}
\]

or

\[
y(\Omega) = X(\Omega). \quad \square
\] (9)

**Corollary 1.** By modifying the impulse response of the CT_LTV filter to \( h_{WFT}(t, \tau) = e^{-2j\Omega(t-\tau)} \), where

\[
\text{Fig. 1. (a) Modulated filtering interpretation of STFT as in (5), (b) as in (6).}
\]
frequency scale

\[ y(t) = X(f(t)) \quad -\infty \leq t \leq \infty, \quad \Rightarrow \quad y(\Omega) = X(f(\Omega)). \]

Any desired warping of the frequency domain can be obtained by choosing \( f(t) \).

**Corollary 2.** Using a CT_LTV system with impulse response \( h_{\text{ITFT}}(t, z) = (1/2\pi) e^{j(t-z)} \), we can interpret the IFT as the output \( z(t) \) for an input \( y(t) = X(\Omega)|_{\Omega=t} \).

**Proof.**
\[
z(t) = \int_{-\infty}^{\infty} h_{\text{ITFT}}(t, t-\tau)y(\tau) \, d\tau
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\tau} X(\tau) \, d\tau
= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau) e^{j\tau} \, d\tau
= x(t) \quad \text{(by definition).} \]

**Proposition 2** (The frequency samples of the STFT (TV spectrum), can be interpreted as the output of a CT_LTV filterbank (FB) as shown in Fig. 3. Let the TV impulse response of \( k \)th filter be \( h_k(t, z) = e^{-j\Omega_k(t-z)} w_k(z) \), where \( w_k(t) \) is the window function. The output \( y_k(t) \) is the \( k \)th frequency sample of the STFT of \( x(t) \) with \( w_k(t) = w(t) \) for \( 1 \leq k \leq K \).

**Proof.** Using the \( k \)th filter impulse response, we get the \( k \)th output as follows:
\[
y_k(t) = \int_{-\infty}^{\infty} h_k(t, t-\tau)x(\tau) \, d\tau
= \int_{-\infty}^{\infty} e^{-j\Omega_k(t-\tau)} w(t-\tau)x(\tau) \, d\tau
= \int_{-\infty}^{\infty} x(\tau) w(t-\tau) e^{-j\Omega_k \tau} \, d\tau
= X(t, \Omega_k) \quad \text{(by definition; see (5)).} \]

**Corollary 3.** We can extend the CT_LTV FB to TV multi-resolution FB. Let us denote the bandwidth of \( W_k(\Omega) \) by \( \Delta \Omega_k \). In lieu of uniform frequency sampling

(i.e., \( \Omega_k - \Omega_{k-1} = \text{constant} \), we can choose any arbitrary non-uniform sampling on the frequency axis (such as mel frequency spacing in speech applications [7]). Similarly, \( \Delta \Omega_k \) can also be chosen as a function of frequency \( \Omega_k \) (for example, constant \( Q \) bandwidth for mel frequency spacing). With this kind of choice for \( \Omega_k \) and \( \Delta \Omega_k \) multi-resolution analysis is realized through a TV FB.

### 3. Discrete-time case

The \( N \) point WDFT of a sequence \( x[n] \), \( 0 \leq n \leq N - 1 \), is given by the samples of \( X(z) \) (Z-transform of \( x[n] \)) at \( N \) arbitrary but distinct points on the unit circle [5]. When these samples are uniformly spaced we define it as the \( N \) point DFT of \( x[n] \). Thus, the DFT and IDFT are stated as

\[
X[k] = \sum_{n=0}^{N-1} x[n] W_{kn}, \quad 0 \leq k \leq N - 1; \tag{12}
\]

\[
x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W^{-kn}, \quad 0 \leq n \leq N - 1, \tag{12}
\]

where \( W = e^{-j2\pi/N} \). The DFT operation can also be viewed as an LTI filterbank operation [1], where the DFT coefficients are obtained as the output of a bank of filters. For discrete signals with TV properties, the short-time discrete-time Fourier transform (ST_DTFT) is a well-known tool for generating TV spectrum of the signal. The ST_DTFT \( X[n, \omega] \) of a signal \( x[n] \) is given by

\[
X[n, \omega] = \sum_{m=-\infty}^{\infty} x[m] W[n-m] e^{-j\omega m}, \tag{13}
\]

where \( W[n] \) is a localizing window. Similar to the continuous-time case, for any \( \omega = \omega_1 \) the

\[ y(t) = h(t, \alpha). \]

**Fig. 2.** Block diagram of CT_LTV system.

\[ x(t) \]

\[ h(t, \alpha) \quad y(t) \]

**Fig. 3.** LTV filter interpretation of STFT.
ST_DTFT can be viewed as the output of a modulated filter [4].

Extending the principle of super-position, from the LTI systems, the output \( y[n] \) of a discrete-time linear time-varying (DT_LTV) system is related to the input \( x[n] \) in the following way [6]:

\[
y[n] = \sum_{p=-\infty}^{\infty} h[n, n-p] x[p],
\]

(14)

where \( h[n, m] \) is the TV impulse response of the DT_LTV system and the TV transfer function is given by \( H(n, \omega) = \sum_{m=-\infty}^{\infty} h[n, m] e^{-j\omega m} \).

Similar to continuous-time case, propositions can also be made regarding the DT_LTV system interpretation of WDFT, DFT, and IDFT, which also leads to DCT, DST, etc. 

**Proposition 3** (The DFT can be interpreted as the output of a DT_LTV system). Consider a DT_LTV system with impulse response \( h_{DTFT}[n, m] = e^{-j(2\pi/N)n(n-m)} \) for \( 0 \leq n \leq N - 1 \). If the input to this system is the signal \( x[n] \), the output signal of the system is the \( N \)-pt DFT of \( x[n] \), i.e., \( y[n] = X[n] \) or \( y[k] = X[k] \) for \( 0 \leq k \leq N - 1 \).

**Proof.**

\[
y[n] = \sum_{p=-\infty}^{\infty} h_{DTFT}[n, n-p] x[p]
\]

\[
= \sum_{p=0}^{N-1} e^{-j(2\pi/N)n(n-p)} x[p]
\]

\[
= \sum_{p=0}^{N-1} e^{-j(2\pi/N)n p} x[p]
\]

\[
= X[n] \quad \text{(using (12))}
\]

or

\[
y[k] = X[k].
\]

(15)

i.e., the DFT coefficients are obtained as successive samples of DT_LTV filter output. □

**Corollary 4.** By choosing \( h_{WDFT}[n, m] = e^{-j\psi[n](n-m)} \), \( 0 \leq n \leq N - 1 \), where \( \psi[n] \) is arbitrary, the DT_LTV output can be made to correspond to the WDFT; i.e., the output \( y[n] \) is WDFT of \( x[n] \).

\[
y[n] = \sum_{p=0}^{N-1} e^{-j(2\pi/N)\psi[n](n-p)} x[p]
\]

\[
= \sum_{p=0}^{N-1} e^{-j(2\pi/N)\psi[n] p} x[p],
\]

\[
y[n] = X[\psi[n]]
\]

or

\[
y[k] = X[\psi[k]].
\]

**Corollary 5.** Consider a DT_LTV system with impulse response \( h_{DTFT}[n, m] = (1/N) e^{j(2\pi/N)n(n-m)} \).

If the input to this system is the sequence of DFT coefficients of \( x[n] \), i.e., \( \{X[n]\}_{n=0}^{N-1} \), the output sequence of the system is the sequence \( \{x[n]\}_{n=0}^{N-1} \) (i.e., IDFT of \( X[k] \) for \( 0 \leq k \leq N - 1 \)).

\[
y[n] = \sum_{p=0}^{N-1} \frac{1}{N} e^{j(2\pi/N)n(n-p)} X(e^{j(2\pi/N)p})
\]

\[
= \sum_{p=0}^{N-1} \frac{1}{N} e^{j(2\pi/N)p} X(e^{j(2\pi/N)p})
\]

\[
y[n] = x[n].
\]

**Corollary 6.** The 1-D discrete cosine transform (1D DCT) \( X_{DCT}[k] \) of a sequence \( x[n] \), \( 0 \leq n \leq N - 1 \), is defined as

\[
X_{DCT}[k] = \pi[k] \sum_{n=0}^{N-1} x[n] \cos \left( \frac{(2n+1)k}{2N} \right),
\]

\[
k = 0, 1, \ldots, N - 1,
\]

(17)

where \( \pi[k] \) is defined as

\[
\pi[k] = \begin{cases} 
\sqrt{\frac{1}{N}} & \text{for } k = 0, \\
\sqrt{\frac{2}{N}} & \text{for } k = 1, 2, \ldots, N - 1.
\end{cases}
\]

(18)

These DCT coefficients can be interpreted as the output of a DT_LTV filter with the impulse response \( h_{DCT}[n, m] = (\pi[n/2N](2n+1)) \) for \( n = 0, \ldots, N - 1 \), where \( \pi[n] \) is given by (18). The output \( y[n] \), \( 0 \leq n \leq N - 1 \), of this DT_LTV system with input \( x[n] \), \( 0 \leq n \leq N - 1 \) is

\[
y[n] = \sum_{m=0}^{N-1} h[n, n-m] x[m]
\]

\[
= \sum_{m=0}^{N-1} \pi[n] \cos \left( \frac{\pi n}{2N} \right) x[m]
\]

\[
= \pi[n] \sum_{m=0}^{N-1} x[m] \cos \left( \frac{(2m+1)n}{2N} \right)
\]

\[
y[n] = X_{DCT}[n] \quad \text{(by definition)}.
\]

(19)

Similar results can be shown for 1-D discrete sine transform (1-D DST), inverse DCT and inverse DST.
Proposition 4 (The frequency samples of the TV spectrum, given by ST_DTFT, can be interpreted as the output of a DT_LTV FB). Consider a DT_LTV FB with the $k^{th}$ filter impulse response given by $h_k[n,m] = e^{-j\omega_k(n-m)}w_k[m]$ for $k = 0, 1, \ldots, N - 1$. The $k^{th}$ output $y_k[n]$, $0 \leq k \leq N - 1$, is the $k^{th}$ frequency sample of the ST_DTFT of $x[n]$ with $\omega_k = (2\pi/N)k$ and $w_k[m] = w[m] \forall k$.

Proof. Proof is similar to that of Proposition 2. 

Multi-resolution filterbank analysis can also be interpreted as a DT_LTV FB operation by appropriately choosing $\omega_k$ and $w_k[m]$.

4. Discussion

When the FT of a signal is treated as the output of a system with the signal as input, it is easy to see that the system should be linear, but not time-invariant; i.e., if the input signal is delayed by a certain duration, the output is not delayed by the same amount, but rather it gets multiplied by a phase term, i.e., gets modulated. Therefore, the impulse response of such a system becomes TV and hence the LTV interpretation. We have shown that the FT and its variants, including STFT, can be interpreted as LTV filter operations.

To get some insight into the TV filters corresponding to the above propositions, let us consider the impulse response of the CT_LTV filter in Corollary 1, i.e., $h_{\text{WFT}}(t,z) = e^{-j\Omega(t-z)}$. It can be seen from Proposition 1 and Corollary 2 that $f(t) = t$ and $f(t) = -t$ result in the impulse responses of the CT_LTV filters corresponding to the FT and IFT. The transfer function of the TV filter $h_{\text{WFT}}(t,z)$ is given by

\[
H_{\text{WFT}}(t,\Omega) = \int_{-\infty}^{\infty} h_{\text{WFT}}(t,z) e^{-j\Omega z} \, dz \\
= \int_{-\infty}^{\infty} e^{-j\Omega(t-z)} e^{-j\Omega z} \, dz \\
= e^{-j\Omega t} \left\{ \int_{-\infty}^{\infty} e^{-j\Omega(\Omega-f(t))} \, dz \right\} \\
= 2\pi e^{-j\Omega^2 \delta(\Omega-f(t))},
\]

i.e., at a particular time ‘t’, the TV filter acts as a tuner, selecting a particular frequency of the input signal determined by the function $f(t)$. Any desirable warping of the frequency scale can be realized by appropriately choosing $f(t)$ in the LTV impulse response. This has resulted in the WFT as the output of TV filter. Similar interpretations can also be given to the DT_LTV filter $h_{\text{WFT}}[n,m] = e^{-j\Omega[n-m]}/C_{0}$.

Further, LTV filter is not invertible in general [8], but from Proposition 1 we see that for the specific cases of FT and IFT, the respective TV filters are inverse of each other. Similarly, from Proposition 3 we see that the TV filters for DFT and IDFT are inverse of each other; other inverse filters of similar structure can be constructed. It is also important to note that DTFT of a discrete-time signal could not be interpreted as a filtering operation. This is because the filter interpretation of DTFT will require the input signal to be discrete but the output signal to be continuous!

As shown in (5) and (6), a frequency slice of STFT itself is interpreted as the output of a modulated filterbank (Fig. 1). Since, modulation is not an LTI filtering operation, $X(t,\Omega)$ is not the output of an LTI system for input $x(t)$. Therefore, strictly, the modulated FB is not LTI. In Proposition 2, we have shown that the frequency samples of STFT is the output of a CT_LTV FB where $k^{th}$ filter has a TV frequency response $H_k(t,\Omega) = e^{-j\Omega t}W_k(\Omega - \Omega_k)$, where $w_k(z) = F_w W_k(\Omega)$ is a low-pass filter. Only the phase of $H_k(t,\Omega)$ is TV and this is contributing to the modulation operation as in (6). This is also true for ST_DTFT. Thus, the short-time spectrum can be interpreted as the output of a TV FB.

5. Conclusion

We have shown a relation between the Fourier transform and its variants and the concept of time-varying filter. Although the modulated LTI filterbank interpretation of DFT and modulated FB interpretation of STFT are popular and useful, there is no such filter interpretation of continuous time FT, IFT, warped transform, etc. We have interpreted all these transforms (for both continuous and discrete time) in a common framework of LTV filter operation.

References


