Multirate digital signal processing

Prasanta Kumar Ghosh
Oct24, 2017
Implementation of Sampling Rate Conversion
Polyphase filter structure for efficient implementation of sampling rate converters

\[ H(z) = \cdots + h(0) + h(M)z^{-M} + \cdots \]
\[ \cdots + h(1)z^{-1} + h(M+1)z^{-(M+1)} + \cdots \]
\[ \vdots \]
\[ \cdots + h(M-1)z^{-(M-1)} + h(2M-1)z^{-(2M-1)} + \cdots \]

\[ H(z) = \left[ \cdots + h(0) + h(M)z^{-M} + \cdots \right] \]
\[ + z^{-1}\left[ \cdots + h(1) + h(M+1)z^{-M} + \cdots \right] \]
\[ \vdots \]
\[ + z^{-(M-1)}\left[ \cdots + h(M-1) + h(2M-1)z^{-M} + \cdots \right] \]

\[ H(z) = \sum_{i=0}^{M-1} z^{-i} P_i(z^M) \]
\[ P_i(z) = \sum_{n=-\infty}^{\infty} h(nM + i)z^{-n} \]

M-component polyphase decomposition

Polyphase components

\[ p_i(n) = h(nM + i), \quad i = 0, 1, \ldots, M-1 \]
Polyphase filter structure

\[ Y(z) = H(z)X(z) \]

\[ = P_0(z^3)X(z) + z^{-1}P_1(z^3)X(z) + z^{-2}P_2(z^3)X(z) \]

\[ = P_0(z^3)X(z) + z^{-1}\{P_1(z^3)X(z) + z^{-1}[P_2(z^3)X(z)]\} \]

\[ M = 3. \]
Polyphase filter structure

\[ Y(z) = H(z)X(z) \]

\[ = P_0(z^3)X(z) + z^{-1}P_1(z^3)X(z) + z^{-2}P_2(z^3)X(z) \]

\[ = P_0(z^3)X(z) + z^{-1}\{P_1(z^3)X(z) + z^{-1}[P_2(z^3)X(z)]\} \]

\[ M = 3. \]
Interchange of filters and downsamplers/upsamplers

Noble identities
Interchange of filters and downsamplers/upsamplers

Noble identities

\[ y(n) = x(nD) \leftrightarrow Y(z) = \frac{1}{D} \sum_{i=0}^{D-1} X(z^{1/D} W_D^i) \quad W_D = e^{-j2\pi/D} \]
Interchange of filters and downsamplers/upsamplers

Noble identities

For the first system

\[ Y(z) = \frac{1}{D} \sum_{i=0}^{D-1} V_1(z^{1/D}W_D^i) \]

\[ = \frac{1}{D} \sum_{i=0}^{D-1} H(zW_D^i)X(z^{1/D}W_D^i) \]

\[ V_1(z) = H(z^D)X(z) \]

But

\[ W_D^i = 1 \]

\[ Y(z) = \frac{1}{D} H(z) \sum_{i=0}^{D-1} X(z^{1/D}W_D^i) = H(z)V_2(z) \]
Interchange of filters and downsamplers/upsamplers

Noble identities
Interchange of filters and downsamplers/upsamplers

Noble identities

\[
y(n) = \begin{cases} 
  x\left(\frac{n}{I}\right), & n = 0, \pm I, \pm 2I, \ldots \\
  0, & \text{otherwise}
\end{cases}
\]
\[ \leftrightarrow \quad Y(z) = X(z^I) \]
Interchange of filters and downsamplers/upsamplers

Noble identities

For the first system

\[ Y(z) = H(z^l) V_1(z) = H(z^l) X(z^l) \]

For the second system

\[ Y(z) = V_2(z^l) = H(z^l) X(z^l) \]

It is possible to interchange the LTI filtering and downsampling or upsampling if we properly modify the system function of the filter.
Sampling rate conversion with cascaded integrator comb filters

\[ H(z) = \sum_{k=0}^{M-1} z^{-k} = \frac{1 - z^{-M}}{1 - z^{-1}} \]

Cascaded integrator comb (CIC) filter

Comb filter

Does not require any multiplication or storage for the filter coefficients

Sampling rate conversion with cascaded integrator comb filters

To improve the lowpass frequency response required for sampling rate conversion, we can cascade $K$ CIC filters. As above all integrations can be done before downsampling and all difference operations after downsampling.

Sampling rate conversion with cascaded integrator comb filters

\[ x(n) \xrightarrow{\uparrow I} 1 - z^{-I} \xrightarrow{\frac{1}{1 - z^{-1}}} y(m) \]

\[ x(n) \xrightarrow{1 - z^{-1}} \xrightarrow{\uparrow I} \xrightarrow{\frac{1}{1 - z^{-1}}} y(m) \]

Polyphase structure for decimation and interpolation filters

**decimation**

Why compute filter output and then throw away samples?

Downsampling commutes with addition
Polyphase structure for decimation and interpolation filters

decimation

With noble identity we get
Polyphase structure for decimation and interpolation filters

decimation

Only needed samples are computed and all multiplication and additions are performed at lower sampling rate
Polyphase structure for decimation and interpolation filters

decimation

Commutator model

Commutator rate $= F_x$

$F_x$

$x(n)$

$P_0(z)$

$P_1(z)$

$P_{D-1}(z)$

$F_y = F_x / D$

$y(m)$
Polyphase structure for decimation and interpolation filters
interpolation

\[ x(n) \rightarrow I \rightarrow v(m) \rightarrow H(z) \rightarrow y(m) \]
Polyphase structure for decimation and interpolation filters

interpolation
Polyphase structure for decimation and interpolation filters

interpolation

\[ u_1, u_2, u_3, \ldots \quad u_1, 0, 0, u_2, 0, 0, u_3, \ldots \]

\[ v_1, v_2, v_3, \ldots \quad v_1, 0, 0, v_2, 0, 0, v_3, \ldots \]

\[ w_1, w_2, w_3, \ldots \quad w_1, 0, 0, w_2, 0, 0, w_3, \ldots \]
Polyphase structure for decimation and interpolation filters

Interpolation

Commutator model

\[ F_x \]
\[ x(n) \]
\[ P_0(z) \]
\[ P_1(z) \]
\[ \vdots \]
\[ P_{l-1}(z) \]

Commutator rate = \( I F_x \)

\[ F_y = I F_x \]

\[ y(m) \]
Structures for rational sampling rate conversion $I/D$

Polyphase interpolation by a downsampler. But no need for computing all $I$ interpolated values as only one in $D$ outputs are kept.

$T_{\text{block}} = 3T_x = 5T_y = 15T$

- $T_x$
- $T_y$
- $T$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>...</th>
<th>15</th>
<th>...</th>
<th>20</th>
<th>...</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>...</td>
<td>15</td>
<td>...</td>
<td>20</td>
<td>...</td>
<td>25</td>
</tr>
<tr>
<td>$m$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>...</td>
<td>15</td>
<td>...</td>
<td>20</td>
<td>...</td>
<td>25</td>
</tr>
<tr>
<td>$k_m$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>...</td>
<td>15</td>
<td>...</td>
<td>20</td>
<td>...</td>
<td>25</td>
</tr>
<tr>
<td>$i_m$</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>13</td>
<td>...</td>
<td>19</td>
<td>...</td>
<td>23</td>
</tr>
</tbody>
</table>

Polyphase subfilter index
Sampling rate conversion for bandpass signals is achieved by finding an equivalent lowpass signal, in general.
Sampling rate conversion by an arbitrary factor

What if $I/D = 1023/511$? Or the exact factor is not known when the rate converter is designed? Or the actual rate may not be rational fraction of the input rate?

Polyphase interpolator

Consider polyphase interpolator with $I$ subfilters. It generates samples with spacing $T_x/I$. If this spacing is too small

1. such that the signal values changes by less than quantization step, then the value at $t = nT_x + \Delta T_x$, $0 \leq \Delta \leq 1$ can be approximated by nearest-neighbor (zero-order hold interpolation)

2. two point linear interpolation can be performed

$$y(nT_x + \Delta T_x) = (1 - \Delta)x(n) + \Delta x(n + 1)$$