E1 216 COMPUTER VISION

LECTURE 02: CAMERA GEOMETRY

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2024

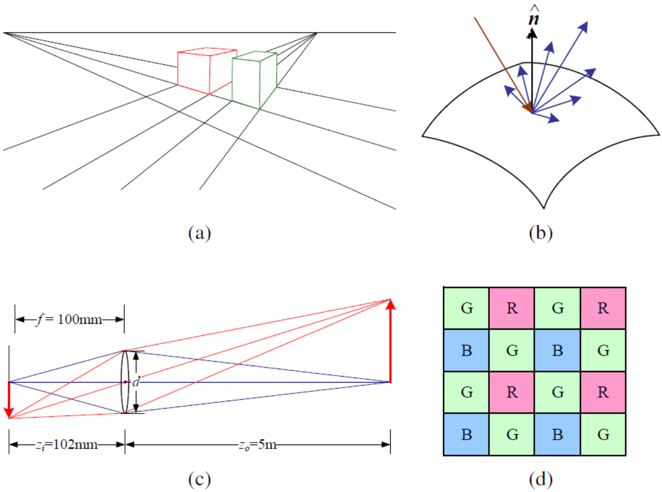
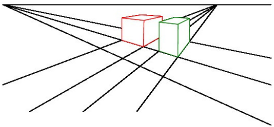
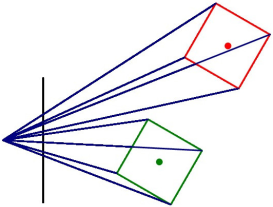


Figure 2.1 *A few components of the image formation process: (a) perspective projection; (b) light scattering when hitting a surface; (c) lens optics; (d) Bayer color filter array.*

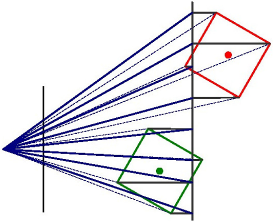
Projection Models



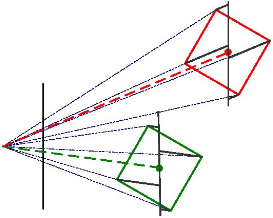
(a) 3D view



(e) perspective



(b) orthography



(d) para-perspective

Camera Geometry

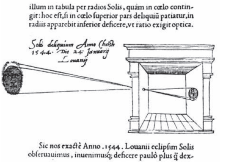
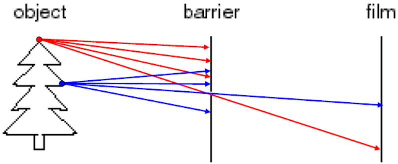
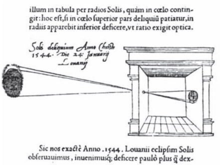


FIGURE 1.1. The camera obscura was used by Reinerus Gemma-Frisius in 1544 to observe an eclipse of the sun.



How do we capture light?

Science for the Curious Photographer, Steve Seitz



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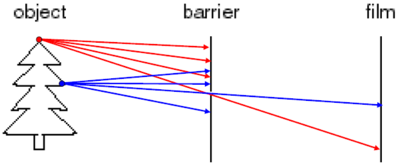
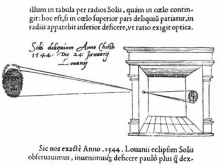


How do we capture light?

Pinhole Camera

Science for the Curious Photographer, Steve Seitz

Camera Geometry



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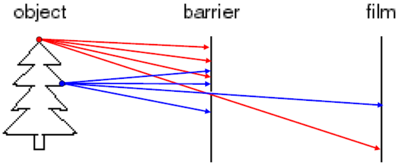

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Pinhole Camera

Why?

Science for the Curious Photographer, Steve Seitz

What is a Camera?

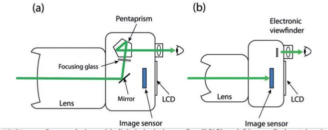
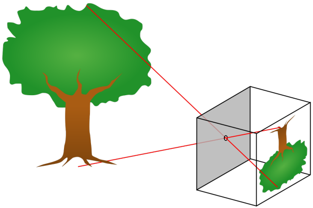


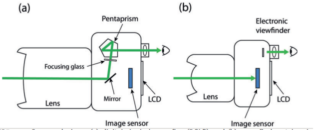
FIGURE 3.4. Camera designs: (a) digital single lens reflex (DSLR) and (b) rangefinder style mirrorless camera.



Camera = Pinhole

Science for the Curious Photographer, wikipedia

What is a Camera?



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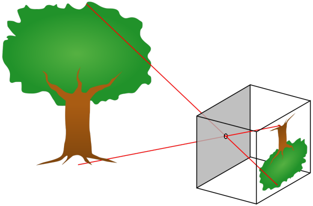

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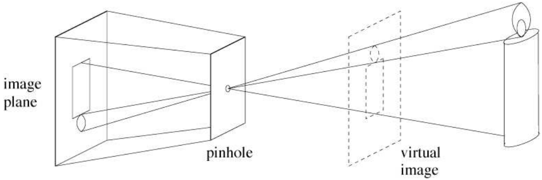
Powerful Mathematical Model

Science for the Curious Photographer, wikipedia

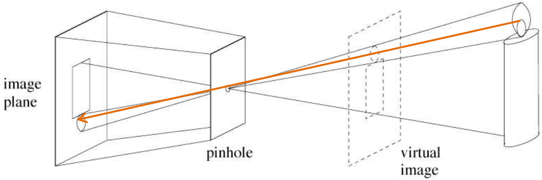
Camera Geometry



Pinhole Camera Model

- What are the consequences of this model?
- Imagine you project a 3D point onto the image plane
- Where did it come from?

Camera Geometry



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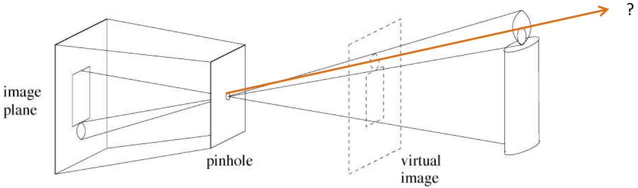

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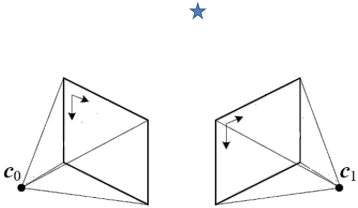
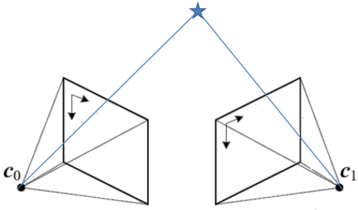


Recovering 3D Geometry

- Consider two cameras (one is never enough)
- Take pictures
- Maps to points on image planes
- Know linear constraint on 3D point from left camera
- Use right camera constraint to intersect

Camera Geometry



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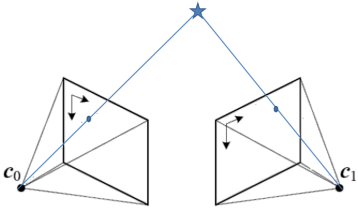

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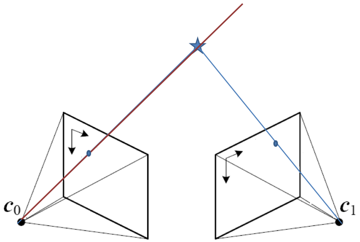

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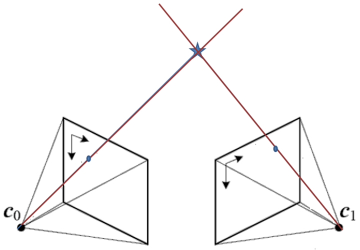

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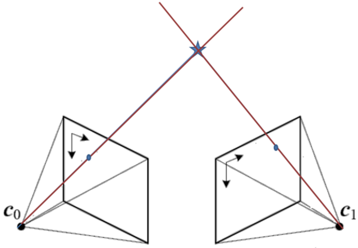
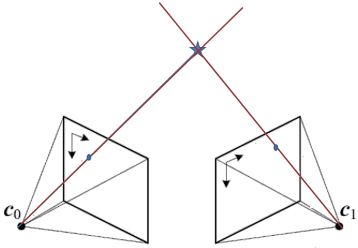


Many Considerations

- Do we know camera parameters? (intrinsic calibration)
- Do we know orientations of cameras? (extrinsic calibration)
- Match features (representation,matching,robustness)
- Do the backprojected rays intersect? (structure estimation)
- Extend this principle to multiple images
- Non-trivial, but many important advances
- State-of-the-art can handle large datasets ($> 10^4$ images)

Camera Geometry



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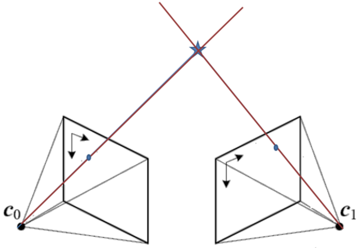

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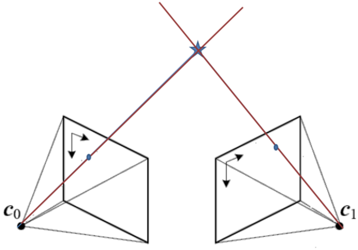

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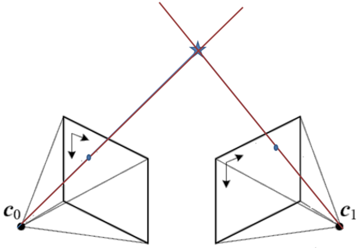

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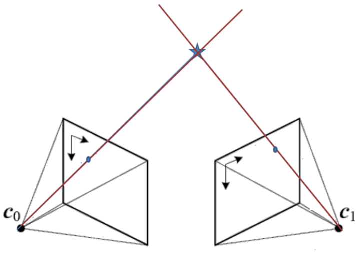

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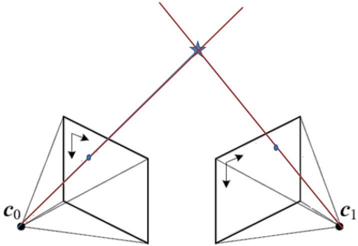

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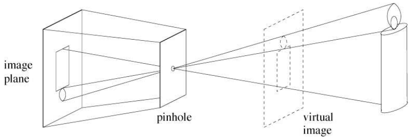
What's a Good Camera Model?



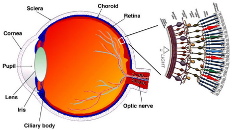
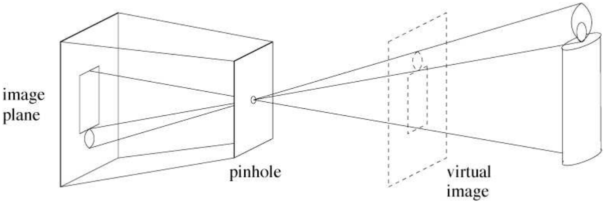


Fig. 1.1. A drawing of a section through the human eye with a schematic enlargement of the retina.

Camera Systems

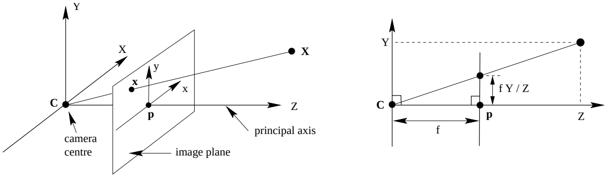
- Camera imaging surface - typically a rectangular plane
- Human retina is closer to a spherical surface
- Vastly different image plane geometries
- Fundamental 3D-2D imaging model is the same
- Spatial sampling is uniform for typical cameras
- Omnidirectional cameras

Camera Model : Perspective Projection



- Very simple geometry
- Sufficiently powerful representation
- *Virtual Image* considered in front of focus
- Real cameras *do* deviate from this model

Camera Model : Perspective Projection

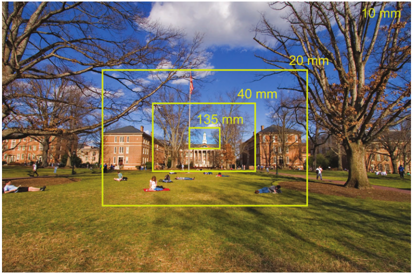


- Coordinate system with origin at camera centre
- World coordinates of point $\boldsymbol{P} = (\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$
- Image projection measured in *local* image coordinate system
- Image coordinates $\boldsymbol{p} = (\boldsymbol{x}, \boldsymbol{y})$

By simple similarity of triangles we have

$$\boldsymbol{x} = \frac{f\boldsymbol{X}}{\boldsymbol{Z}}$$
$$\boldsymbol{y} = \frac{f\boldsymbol{Y}}{\boldsymbol{Z}}$$

Camera Model : Perspective Projection



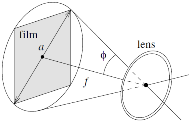


FIGURE 1.9: The field of view of a camera. It can be defined as 2ϕ, where $\phi \stackrel{\text{def}}{=} \arctan \frac{a}{2f}$, a is the diameter of the sensor (film, CCD, or CMOS chip), and f is the focal length of the camera.

Changing focal length

- Keep camera fixed, change focal length
- What happens to the volume imaged?

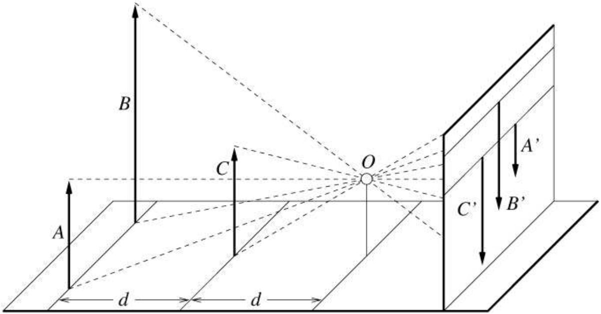
Science for the Curious Photographer; Forstyh and Ponce 2nd Edition.

Camera Model : Perspective Projection

$$x = \frac{fX}{Z}$$
$$y = \frac{fY}{Z}$$

Implications

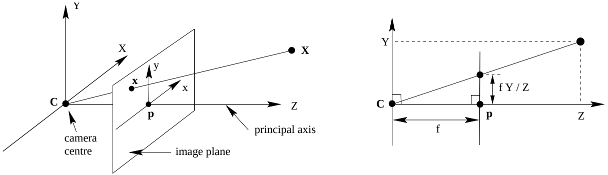
- Different points are scaled different according to depth
- Introduces non-linearities in the relationships
- Distant objects are smaller
- Cannot judge object size with a single image



Perspective projection

- Cannot judge object size with a single image
- Judgement of size can be wrong!

From twitter.com/rainmaker1973



Two co-ordinate systems!

- Remember that we have two measurements of interest
  - Measurements on the image plane
  - Measurements in the 3D world
- Our interest is to relate the two

Camera Model (contd.)

Consider perspective projection model

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{f}{Z} \begin{bmatrix} X \\ Y \end{bmatrix}$$

Now let's translate the frame of reference (or camera), new co-ordinates

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{f}{Z + t_z} \begin{bmatrix} X + t_x \\ Y + t_y \end{bmatrix}$$

Camera Model (contd.)

Consider perspective projection model

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{f}{Z} \begin{bmatrix} X \\ Y \end{bmatrix}$$

Now let's translate the frame of reference (or camera), new co-ordinates

$$x^{'} = f\frac{(X + t_x)}{(Z + t_z)}$$

$$y^{'} = f\frac{(Y + t_y)}{(Z + t_z)}$$

Camera Model (contd.)

Or if we were to rotate the camera by rotation matrix $\boldsymbol{R}$

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

The new 3D coordinates would be

$$\begin{bmatrix} \boldsymbol{X}' \\ \boldsymbol{Y}' \\ \boldsymbol{Z}' \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{Z} \end{bmatrix}$$

Camera Model (contd.)

Therefore, the new image projections would look like

$$
\begin{aligned}
x &= f\frac{r_{11}X + r_{12}Y + r_{13}Z}{r_{31}X + r_{32}Y + r_{33}Z} \\
y &= f\frac{r_{21}X + r_{22}Y + r_{23}Z}{r_{31}X + r_{32}Y + r_{33}Z}
\end{aligned}
$$

- Now if we apply an additional transformation, the two rotations would get entangled
- End result of multiple transformations is very messy!
- Need a cleaner approach

Homogeneous Representations

To arrive at a solution, we take recourse to geometry

Geometric approaches

- "Purist" view - co-ordinate free approach to geometry
- Classical theorems due to Euclid
- Since Descartes, there's an algebraic view of geometric constructs
- Duality : Geometry $\leftrightarrow$ Algebra
- Circle : Centre + Radius $\leftrightarrow (\boldsymbol{p} - \boldsymbol{p}_0)^T (\boldsymbol{p} - \boldsymbol{p}_0) = r^2$

Homogeneous Representations

Consider a line $y = mx + c$
Rewrite as $mx - y + c = 0$
or generally as
ax+by+c = 0

Rewriting this we have

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

Homogeneous Representation of a Line

$$\underbrace{\begin{bmatrix} a & b & c \end{bmatrix}}_{\boldsymbol{l}} \overbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}^{\boldsymbol{p}} = 0$$

this results in a nice symmetric form

$$\boldsymbol{l}^T \boldsymbol{p} = 0$$

This form has many advantages over $y = mx + c$ form

Homogeneous Representation of a Line

Consider the intersection of two lines
To solve for the point of intersection

$$y = m_1 x + c_1$$
$$y = m_2 x + c_2$$

Solve simultaneous equations by substitution, $x = \frac{(y - c_1)}{m_1}$

$$\begin{aligned} y &= (y - c_1)\frac{m_2}{m_1} + c_2 \\ (1 - \frac{m_2}{m_1})y &= c_2 - \frac{c_1 m_2}{m_1} \\ y &= = \frac{(c_2 - \frac{c_1 m_2}{m_1})}{(1 - \frac{m_2}{m_1})} \end{aligned}$$

Quite a mess!!

Homogeneous Representation of a Line

In the homogeneous system of representation we have

$$\boldsymbol{l}_1{}^T \boldsymbol{p} = 0$$
$$\boldsymbol{l}_2{}^T \boldsymbol{p} = 0$$

Therefore, the co-ordinates of the intersection is given by

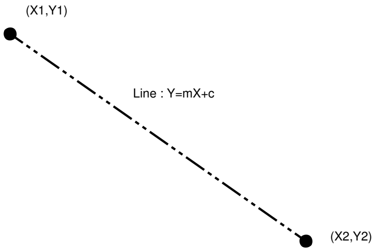
$$\boldsymbol{p} = \boldsymbol{l}_1 \times \boldsymbol{l}_2$$

Verify

- $\boldsymbol{l}_1{}^T(\boldsymbol{l}_1 \times \boldsymbol{l}_2) = 0$
- $\boldsymbol{l}_2{}^T(\boldsymbol{l}_1 \times \boldsymbol{l}_2) = 0$
- Much cleaner way of solving

Homogeneous Representation of a Line

Consider the line through two given points



Usual solution is messy
Instead, using homogeneous coordinates, we get the dual representation

$$\text{Line} : \boldsymbol{l} = \boldsymbol{p}_1 \times \boldsymbol{p}_2$$

Easily verified that this satisfies the requirements

- $(\boldsymbol{p}_1 \times \boldsymbol{p}_2)^T \boldsymbol{p}_1 = 0$
- $(\boldsymbol{p}_1 \times \boldsymbol{p}_2)^T \boldsymbol{p}_2 = 0$

Homogeneous Representation

The key relationship to note is that

$$\underbrace{\begin{bmatrix} a & b & c \end{bmatrix}}_{\boldsymbol{l}} \overbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}^{\boldsymbol{p}} = 0$$

results in a nice symmetric (and homogeneous) form

$$\boldsymbol{l}^T \boldsymbol{p} = 0$$

This form has many advantages over $y = mx + c$ form

Homogeneous Representation

In homogeneous form everything upto unknown scalar

Homogeneous

$\mathbb{R}^n \mapsto \mathbb{R}^{n+1}$

$$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Inhomogeneous

$\mathbb{R}^n \mapsto \mathbb{R}^{n-1}$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \mapsto \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix}$$

Homogeneous Forms

- Embed in higher dimensions by appending a 1 (canonical)
- Homogeneous forms are equivalent upto scale
- Only ratios matter
- $[u, v, w] = \lambda [u, v, w], \forall \lambda \neq 0$
- Notice $[0, 0, 0]$ is not admissible

Homogeneous Representation

In homogeneous form everything upto unknown scalar

Homogeneous

$\mathbb{R}^n \mapsto \mathbb{R}^{n+1}$

$$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

Inhomogeneous

$\mathbb{R}^n \mapsto \mathbb{R}^{n-1}$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \mapsto \begin{bmatrix} \frac{u}{w} \\ \frac{v}{w} \end{bmatrix}$$

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- Homogeneous forms are equivalent upto scale
- Only ratios matter
- $[u, v, w] = \lambda [u, v, w], \forall \lambda \neq 0$
- Notice $[0, 0, 0]$ is not admissible

7.7 Homogeneous Coordinates

Representing the points of the projective plane $\mathbb{RP}^2$ by lines through O gives *coordinates* to $\mathbb{RP}^2$ via the coordinates (x, y, z) of three-dimensional space. Such coordinates were invented by Möbius (1827) and Plücker (1830), and they are called *homogeneous* because each algebraic curve in $\mathbb{RP}^2$ is expressed by a homogeneous polynomial equation $p(x, y, z) = 0$. The simplest case is that of a projective line, which, as we saw in Section 7.5, is represented by a plane through O. Its equation therefore has the form

$$ax + by + cz = 0, \quad \text{for some constants } a, b, c, \text{ not all zero.}$$

Such an equation is called *homogeneous of degree* 1, because each nonzero term is of degree 1 in the variables x, y, z.

The *homogeneous coordinates of a point* P in $\mathbb{RP}^2$ are simply the coordinates of *all* points on the line through O that represents P. It follows that

Geometries in Computer Vision

- Geometry : Topological Space + Axioms
- Different set of axioms $\rightsquigarrow$ Different Geometries
  - Euclidean (Distances and Angles)
  - Affine (Parallelism)
  - Projective (Straight Line)
  - Non-linear (Riemannian Manifolds)

Stratification of transform space

Euclidean $\subset$ Affine $\subset$ Projective

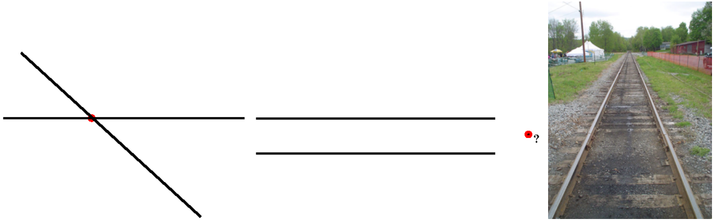
Euclidean Geometry

Axioms of incidence

- Familiar concepts from Euclidean geometry
- Length is a fundamental property of Euclidean Geometry
- Construction with straightedge and compass
- Axioms of Euclid

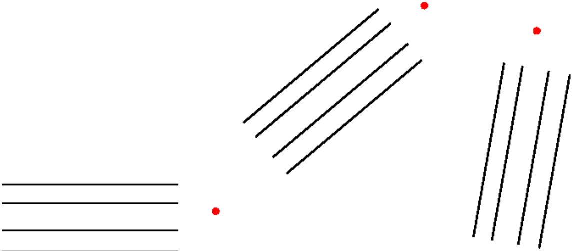
Following Hilbert state the axioms as

- For any two points A, B, a unique line passes through A, B
- Every line contains at least two points
- There exist three points not all on the same line
- **Parallel axiom** : For any line $\mathcal{L}$ and point $\mathcal{P}$ outside $\mathcal{L}$, there is exactly one line through $\mathcal{P}$ that does not meet $\mathcal{L}$



Wikipedia

- Two points have a unique line through them (join)
- Two lines have a unique intersection point (meet)
- What happens when the lines are parallel?
- What does it mean to say that they “intersect at ∞”?



- **Question** : Are all ∞ intersection points the same?
- The answer lies in the geometry of projective space
- Recall homogeneous representations

Homogeneous Forms

Parallel Lines

- Recall line equation: $\boldsymbol{l}^T\boldsymbol{p} = 0$
- $\boldsymbol{l}$ and $\boldsymbol{p}$ upto scale factor $\boldsymbol{l}^T\boldsymbol{p} = (\lambda \boldsymbol{l})^T(\lambda' \boldsymbol{p}) = 0$
- Intersection of two lines $\boldsymbol{p} = \boldsymbol{l}_1 \times \boldsymbol{l}_2$
- When are lines parallel?
- $\boldsymbol{l}_1 = \begin{bmatrix} a & b & c \end{bmatrix}$
- $\boldsymbol{l}_2 = \begin{bmatrix} a & b & c' \end{bmatrix}$
- Intersection point $\boldsymbol{p}$?

Homogeneous Forms

Parallel Lines

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Homogeneous Forms

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Homogeneous Forms

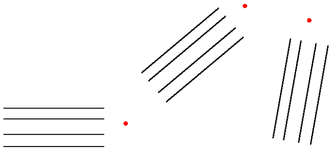
$$\boldsymbol{p} = \boldsymbol{l}_1 \times \boldsymbol{l}_2 = \begin{bmatrix} (c^{'} - c)b & (c - c^{'})a & 0 \end{bmatrix} = [b, -a, 0]$$

What is the inhomogeneous form of $\boldsymbol{p}$?

Parallel Lines

- Recall line equation: $\boldsymbol{l} = \boldsymbol{p}_1 \times \boldsymbol{p}_2$
- $\boldsymbol{l}$ and $\boldsymbol{p}$ upto scale factor
- Intersection of two lines $\boldsymbol{p} = \boldsymbol{l}_1 \times \boldsymbol{l}_2$
- When are lines parallel?
- $\boldsymbol{l}_1 = \begin{bmatrix} a & b & c \end{bmatrix}$
- $\boldsymbol{l}_2 = \begin{bmatrix} a & b & c^{'} \end{bmatrix}$
- Intersection point $\boldsymbol{p}$?

Homogeneous Forms



$$p = l_1 \times l_2 = \begin{bmatrix} (c' - c)b & (c - c')a & 0 \end{bmatrix}$$
$$= [b, -a, 0]$$

What is the inhomogeneous form of p?
Distinct "points at infinity"

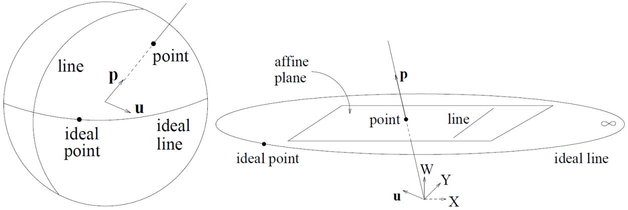
Parallel Lines

- Recall line equation: $l = p_1 \times p_2$
- l and p upto scale factor
- Intersection of two lines $p = l_1 \times l_2$
- When are lines parallel?
- $l_1 = \begin{bmatrix} a & b & c \end{bmatrix}$
- $l_2 = \begin{bmatrix} a & b & c' \end{bmatrix}$
- Intersection point p?

Projective Geometry

- Represent the projective plane as $\mathbb{P}^2$
- Obtained by adding all ∞ points
- ∞ points form a ‘line at infinity’. Why?
- Got rid of special case of parallel lines
- All lines have a unique intersection now
- So what is this space useful for?

Projective Geometry



(d) $\mathbb{P}^2 \equiv S^2$ (e) $\mathbb{P}^2 \equiv \mathbb{R}^3 \backslash \{\mathbf{0}\} / \simeq$

- Projective plane is topologically equivalent to unit sphere
- Associate with half-sphere to projective scale
- Where is the line at infinity on S^2?
- $\mathbb{P}^2$ is equivalent to $\mathbb{R}^3$ with origin removed, under equivalence relationship of scale

Homogeneous Form

Basic Definition

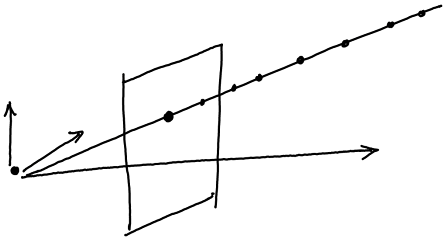
- n-dim real **affine space** is set of all points $(x_1, \cdots, x_n) \in \mathbb{R}^n$
- **Projective space** $\mathbb{P}^n$ given by
  - $(x_1, \cdots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$
  - at least one x_i is non-zero
  - for $\lambda \neq 0$, all $(\lambda x_1, \cdots, \lambda x_n, \lambda x_{n+1})$ are equivalent
- Homogeneous coordinates obtained by $(x_1, \cdots, x_n, 1)$

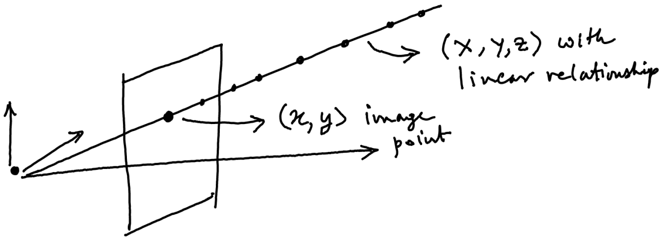
Homogeneous Form

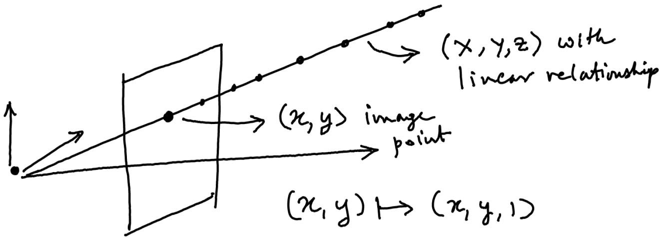
- Let the homogeneous form be $\boldsymbol{X} = (\boldsymbol{X}_1, \cdots, \boldsymbol{X}_{n+1})$
- Let the inhomogeneous form be $\boldsymbol{x} = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$
- Equivalence relationship : $[\boldsymbol{x}, 1] = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, 1) \simeq \boldsymbol{X}$
- $\boldsymbol{x}_i = \frac{\boldsymbol{X}_i}{\boldsymbol{X}_{n+1}}$

Line at Infinity

- **Question**: What is the homogeneous form for points at ∞?
- Is this homogeneous form $[\boldsymbol{x}, 1]$ always valid?
- $[\boldsymbol{x}, 0]$ is also in projective space
- $[\boldsymbol{x}, 0]$ does not have a finite inhomogenous form
- Projective Space: $[\boldsymbol{x}, 1]$ (affine space) $\cup$ $[\boldsymbol{x}, 0]$ (line at ∞)

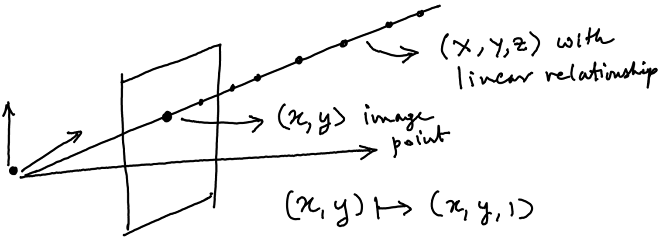



(X, Y, Z) with
linear relationship
(x, y) image
point



$(x, y) \mapsto (x, y, 1)$

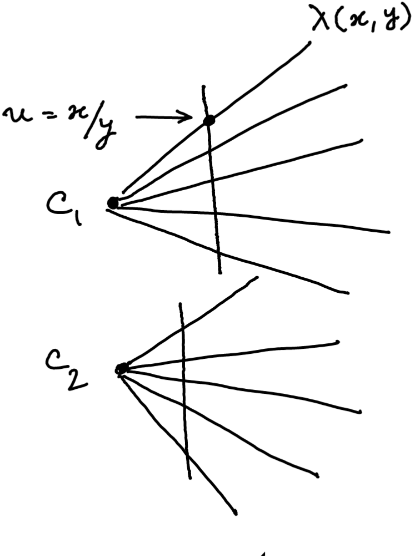
$(x, y, 1)$ is also point in (X, Y, Z)



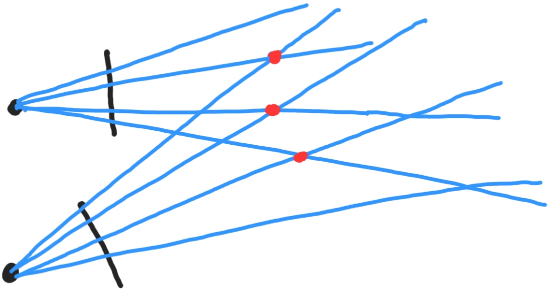
$$(x, y) \mapsto (x, y, 1)$$

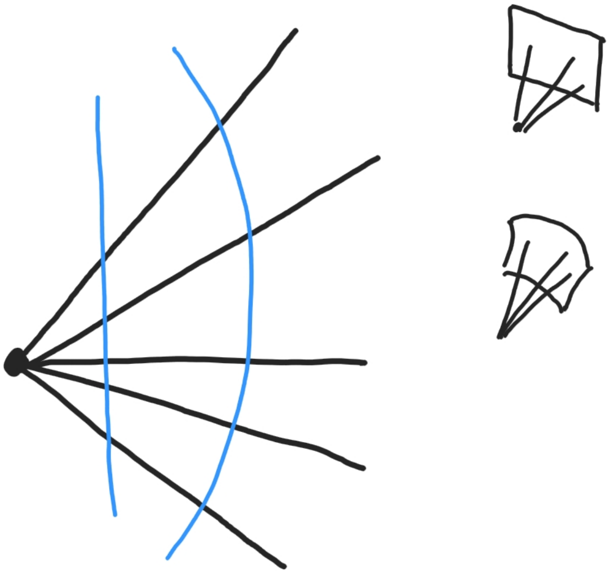
$(x, y, 1)$ is also point in (X, Y, Z)

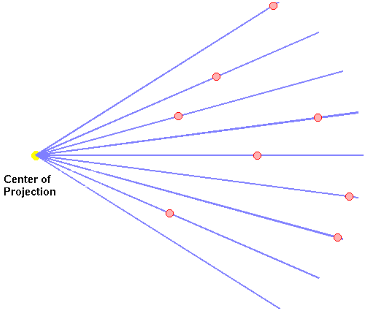
$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$



X(x, y)
u = x/y
C1
C2



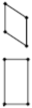
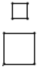




Camera = Camera Centre!

- Consider a centre of projection
- Establishes equivalence classes
- All points on ray are projectively equivalent (beads on wire)
- What happens when they line up?
- Camera model

Transformation Groups

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

$$
\underset{\textit{Projective}}{\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}}
\quad
\underset{\textit{Affine}}{\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ 0 & 0 & 1 \end{bmatrix}}
\quad
\underset{\textit{Euclidean}}{\begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ 0 & 0 & 1 \end{bmatrix}}
$$

Euclidean

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Projective

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Two interpretations

- Euclidean vs. Projective transformations
- $\boldsymbol{H} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ (9 dof)
- $\boldsymbol{H} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ (8 dof)

Since in projective space

$$\mathbb{P}^n, \text{ all } (\lambda x_1, \cdots, \lambda x_n, \lambda x_{n+1})$$

are equivalent, we can *linearise* our imaging model
Recall that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{Z} \begin{bmatrix} X \\ Y \end{bmatrix}$$

For now assume $f = 1$
then by **embedding** image and world points in projective spaces we have

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \frac{1}{Z} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

We now have

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \frac{1}{Z} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Recall, that scaled points are projectively equivalent, i.e.

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\boldsymbol{p} = \boldsymbol{P}$$

We have now managed to linearise the relationship

Projective Geometry

Projective representations for both image and world points

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Euclidean transformation of 3D points

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Projective Geometry

General process of taking a picture

- Apply Euclidean motion to 3D points
- Project onto image plane

Combining two steps we get

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \left[\, R \mid t \, \right] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Projective Geometry

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Taking a Picture

- 3D Point in Homogeneous Form
- Rigid 3D Motion
- Ideal Pinhole Camera
- Image Projection
- $\mathbb{P}^3 \to \mathbb{P}^2$

Projective Geometry

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \boxed{\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}}$$

Taking a Picture

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- $\mathbb{P}^3 \longrightarrow \mathbb{P}^2$

Projective Geometry

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Projective Geometry

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

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$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Taking a Picture

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Projective Geometry

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Taking a Picture

- 3D Point in Homogeneous Form
- Rigid 3D Motion
- Ideal Pinhole Camera
- Image Projection
- $\mathbb{P}^3 \rightarrow \mathbb{P}^2$

Camera Model

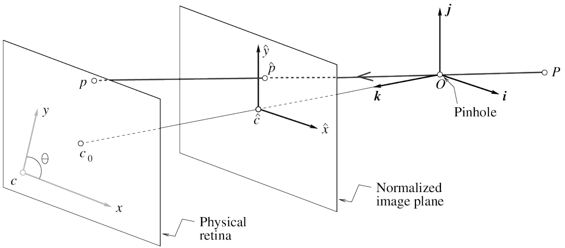


FIGURE 1.14: Physical and normalized image coordinate systems.

Intrinsic Parameters

- Focal length f
- Shift in origin or image center (u_0, v_0)
- Rectangular pixel dimensions (k_u, k_v)
- Imaging plane may be skewed by angle θ

Many deviations from an idealised model

Makes the entire imaging model very messy

Further, the effects of the camera parameters can be represented as a matrix form

$$\boldsymbol{K} = \begin{bmatrix} fk_u & -fk_u cot\theta & -u_0 \\ 0 & \frac{fk_v}{sin\theta} & -v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

General form for the transformation matrix is

$$\begin{bmatrix} fk_u & -fk_u cot\theta & -u_0 \\ 0 & \frac{fk_v}{sin\theta} & -v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & \boldsymbol{t}_x \\ r_{21} & r_{22} & r_{23} & \boldsymbol{t}_y \\ r_{31} & r_{32} & r_{33} & \boldsymbol{t}_z \end{bmatrix}$$

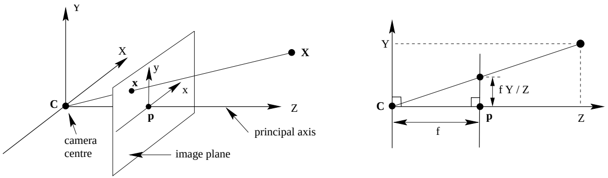
$$\underbrace{\begin{bmatrix} fk_u & -fk_u cot\theta & -u_0 \\ 0 & \frac{fk_v}{sin\theta} & -v_0 \\ 0 & 0 & 1 \end{bmatrix}}_{intrinsic} \overbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix}}^{extrinsic}$$

Put simply the general form of the projective transformation is

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} & \boldsymbol{P}_{13} & \boldsymbol{P}_{14} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} & \boldsymbol{P}_{23} & \boldsymbol{P}_{24} \\ \boldsymbol{P}_{31} & \boldsymbol{P}_{32} & \boldsymbol{P}_{33} & \boldsymbol{P}_{34} \end{bmatrix}$$

Has $3 \times 4 - 1 = 11$ degrees of freedom

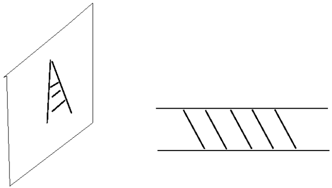
REMINDER!



Two co-ordinate systems!

- Remember that we have two measurements of interest
  - Measurements on the image plane
  - Measurements in the 3D world
- Our interest is to relate the two

Why Projective Geometry?



- A camera is a projective engine
- Simpler representation than affine or Euclidean forms
- Can handle points-at-∞ naturally (fewer special cases)
- Most general representation for our problems

Projective Representations

Reminder

- We are dealing with three types of Projective transformations or mappings
  - Transformations of image plane ($\boldsymbol{H}_{3\times 3} : \mathbb{P}^2 \rightarrow \mathbb{P}^2$)
  - Imaging by a pinhole camera ($\boldsymbol{P}_{3\times 4} : \mathbb{P}^3 \rightarrow \mathbb{P}^2$)
  - Projective change of basis for 3D space ($\boldsymbol{H}_{4\times 4} : \mathbb{P}^3 \rightarrow \mathbb{P}^3$)

Representation of the Projective Camera

$$\boldsymbol{P} = \boldsymbol{K} \left[\begin{array}{ccc|c} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} & \boldsymbol{R}_{13} & \boldsymbol{T}_1 \\ \boldsymbol{R}_{21} & \boldsymbol{R}_{22} & \boldsymbol{R}_{23} & \boldsymbol{T}_2 \\ \boldsymbol{R}_{31} & \boldsymbol{R}_{32} & \boldsymbol{R}_{33} & \boldsymbol{T}_3 \end{array} \right] \quad vs. \quad \left[\begin{array}{ccc|c} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} & \boldsymbol{P}_{13} & \boldsymbol{P}_{14} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} & \boldsymbol{P}_{23} & \boldsymbol{P}_{24} \\ \boldsymbol{P}_{31} & \boldsymbol{P}_{32} & \boldsymbol{P}_{33} & \boldsymbol{P}_{34} \end{array} \right]$$

- Distinction between perspective and projective cameras
- Perspective is a model for a true Euclidean (rigid) transformation
- Perspective camera is a special case of projective camera
- Projective camera is a *purely* mathematical engine
- Projective camera is not necessarily physically realisable
- What about degrees of freedom?

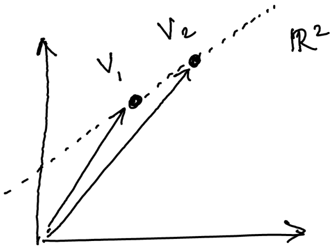
Euclidean Transformation in $\mathbb{R}^3$

$$\begin{aligned} \boldsymbol{P}^{'} &= \boldsymbol{R}\boldsymbol{P} + \boldsymbol{T} \\ \boldsymbol{P}^{'} &= \boldsymbol{R}(\boldsymbol{P} + \boldsymbol{T}) \end{aligned}$$

- First rotate then translate
- Second translate then rotate
- Both are valid representations
- We will prefer the first form over the second
- **Warning : Always understand which one is used!**

EXTRA MATERIAL
NOT PART OF SYLLABUS

Affine Geometry



Consider $v_1, v_2 \in \mathbb{R}^2$
Linear combination: Span $\{v_1, v_2\}$
Affine combination: Line in $\mathbb{R}^2$

Affine Combinations

Consider vectors $v_1, \cdots, v_k \in \mathbb{R}^n$
Affine Combination: $\lambda_1 v_1 + \cdots \lambda_k v_k$
$\lambda_1, \cdots, \lambda_k \in \mathbb{R}$
Restriction: $\lambda_1 + \cdots + \lambda_k = 1$

Linear Combinations

Consider vectors $v_1, \cdots, v_k \in \mathbb{R}^n$
Linear Combination:
$\lambda_1 v_1 + \cdots + \lambda_k v_k \in \mathbb{R}^n$
$\lambda_1, \cdots, \lambda_k \in \mathbb{R}$

Convex Combinations

Restriction: $\lambda_1 + \cdots + \lambda_k = 1$
Further restriction $\lambda_i \geq 0$

Affine Geometry

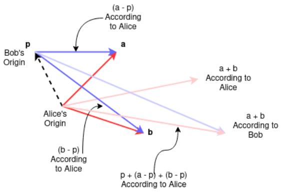
Vector Subspace

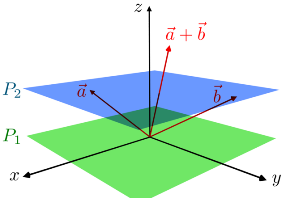
- $A \subseteq \mathbb{R}^n$
- $\mathbf{0} \in A$
- $\boldsymbol{a} \in A \Rightarrow \lambda \boldsymbol{a} \in A$
- $\boldsymbol{a}, \boldsymbol{b} \in A \Rightarrow \boldsymbol{a} + \boldsymbol{b} \in A$
- Points and vectors coincide
- Equipped with inner product
- Distances and angles preserved

Affine Subspace

- $A \subseteq \mathbb{R}^n$
- No origin
- $\boldsymbol{a} \in A \Rightarrow \lambda \boldsymbol{a} \in A$
- $\boldsymbol{a}, \boldsymbol{b} \in A \Rightarrow \lambda \boldsymbol{a} + (1 - \lambda) \boldsymbol{b} \in A$
- $A - \boldsymbol{a}$ is a vector space for any $\boldsymbol{a} \in A$
- Vectors only as differences (translations)
- Only parallelism is preserved

Affine Geometry





Affine Subspaces

- Consider the 2D plane, but forget origin
- What can two independent observers agree upon?
- Second observer assumes that $\boldsymbol{p}$ is the origin
- Adding two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ results in $\boldsymbol{p} + (\boldsymbol{a} - \boldsymbol{p}) + (\boldsymbol{b} - \boldsymbol{p})$
- When linear combination is $\lambda \boldsymbol{a} + (1 - \lambda)\boldsymbol{b}$, observers agree
- Observers know the “affine structure” but not the “linear structure”
- Direction is a fundamental property here, not length

wikipedia