# E1.216 COMPUTER VISION

## LECTURE
## MOTION FIELD AND OPTIC FLOW

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In this lecture we shall look at **differential motion** and **optic flow**

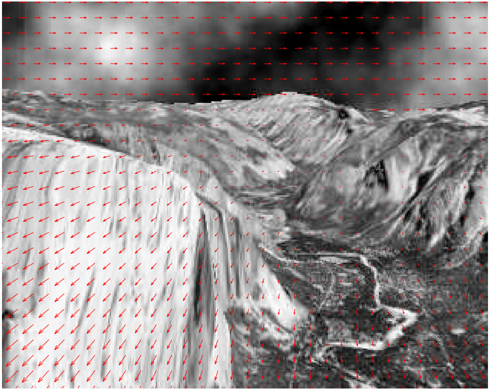
**Warning: Related but different concepts!**

# Motion Field and Optic Flow

## Differential 3D Motion and Estimation

- Motion between two frames is small
- Same epipolar geometry applies as in discrete motion
- Estimation is entirely different due to small motion
- Dense motion field can be estimated
- Structure and camera motion can be derived

# Motion Field and Optic Flow



## Image Flow Field

- Flow field between two images
- Each field vector associated two points in images
- Flow field $\longleftrightarrow$ pointwise correspondence

# Motion Field and Optic Flow

## Image Flow Field

- Image intensity based estimation
- **Image Flow Field $\neq$ Motion Field**
- Image based constraint is ill-posed
- Different additional constraints
- Each constraint results in different estimator
    - Constant flow : Lucas-Kanade algorithm
    - Affine flow
    - Smooth flow : Horn-Schunck algorithm
- There exists a **large** body of literature
- We shall only cover the very elementary issues

# Motion Field and Optic Flow

## 3D Velocity and Image Velocity

- Consider points in 3D space $\boldsymbol{P} = (\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})^T$
- Let this point move in time *relative to camera*
- Instantaneous 3D velocity is

$$\boldsymbol{V} = \frac{d\boldsymbol{P}(\mathbf{t})}{dt} = \left(\frac{d\boldsymbol{X}}{dt}, \frac{d\boldsymbol{Y}}{dt}, \frac{d\boldsymbol{Z}}{dt}\right)^T$$

- Projection model $\boldsymbol{x} = \frac{f\boldsymbol{X}}{\boldsymbol{Z}}$ etc.

Much of this presentation closely follows David Heeger's notes on "Motion Estimation"

## 3D Velocity and Image Velocity

- Instantaneous 3D velocity is

$$\boldsymbol{V} = \frac{d\boldsymbol{P}(\mathbf{t})}{dt} = \left(\frac{d\boldsymbol{X}}{dt}, \frac{d\boldsymbol{Y}}{dt}, \frac{d\boldsymbol{Z}}{dt}\right)^T$$

- 3D trajectory traces out a 2D path on image plane
- Image velocity is

$$\boldsymbol{\theta} = \left(\frac{d\boldsymbol{x}}{dt}, \frac{d\boldsymbol{y}}{dt}\right)^T$$

- In combination, image velocity is given by

$$\boldsymbol{\theta} = \frac{1}{\boldsymbol{Z}}\left(\frac{d\boldsymbol{X}}{dt}, \frac{d\boldsymbol{Y}}{dt}\right)^T + \frac{1}{\boldsymbol{Z}^2}\frac{d\boldsymbol{Z}}{dt}(\boldsymbol{X}(t), \boldsymbol{Y}(t))^T$$

## 3D Velocity and Image Velocity

Assuming a rigid motion between camera and scene

- Camera translation $\mathbf{T} = (T_x, T_y, T_z)^T$
- Camera instantaneous rotation $\mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z)^T$

Instantaneous 3D velocity is given by

$$\left(\frac{d\boldsymbol{X}}{dt}, \frac{d\boldsymbol{Y}}{dt}, \frac{d\boldsymbol{Z}}{dt}\right)^T = -(\boldsymbol{\Omega} \times \boldsymbol{P} + \boldsymbol{T})^T$$

**Why is this true ?**

# Motion Field and Optic Flow

## 3D Velocity and Image Velocity

Two representations

- **3D Velocity**

$$\left(\frac{d\boldsymbol{X}}{dt}, \frac{d\boldsymbol{Y}}{dt}, \frac{d\boldsymbol{Z}}{dt}\right)^T = -(\boldsymbol{\Omega} \times \boldsymbol{P} + \boldsymbol{T})^T$$

- **Image Velocity**

$$\boldsymbol{\theta} = \frac{1}{\boldsymbol{Z}}\left(\frac{d\boldsymbol{X}}{dt}, \frac{d\boldsymbol{Y}}{dt}\right)^T + \frac{1}{\boldsymbol{Z}^2}\frac{d\boldsymbol{Z}}{dt}(\boldsymbol{X}(t), \boldsymbol{Y}(t))^T$$

Can substitute former into latter equation

# Motion Field and Optic Flow

## 3D Velocity and Image Velocity

Results in motion field for rigid scene

$$\boldsymbol{\theta}(x,y) = p(x,y)\boldsymbol{A}(x,y)\mathbf{T} + \boldsymbol{B}(x,y)\boldsymbol{\Omega}$$

where

$$\begin{aligned} p(x,y) &= \frac{1}{\boldsymbol{Z}(x,y)} \\ \boldsymbol{A}(x,y) &= \begin{bmatrix} -f & 0 & x \\ 0 & -f & y \end{bmatrix} \\ \boldsymbol{B}(x,y) &= \begin{bmatrix} \frac{(xy)}{f} & -(f + \frac{x^2}{f}) & y \\ (f + \frac{y^2}{f}) & -\frac{(xy)}{f} & -x \end{bmatrix} \end{aligned}$$

## 3D Velocity and Image Velocity

$$\boldsymbol{\theta}(x, y) = \frac{1}{\boldsymbol{Z}(x, y)} \boldsymbol{A}(x, y)\mathbf{T} + \boldsymbol{B}(x, y)\boldsymbol{\Omega}$$

- First term is translational component
- Depends on depth of 3D point
- Second component is rotational term
- Independent of point depth

# Motion Field and Optic Flow

## 3D Velocity and Image Velocity

$$\boldsymbol{\theta}(x,y) = \frac{1}{\boldsymbol{Z}(x,y)}\boldsymbol{A}(x,y)\mathbf{T} + \boldsymbol{B}(x,y)\boldsymbol{\Omega}$$

Expanding this we have

$$\begin{aligned}
\theta_1 &= \frac{T_z x - T_x f}{Z} - \omega_y f + \omega_z y + \frac{\omega_x x y}{f} - \frac{\omega_y x^2}{f} \\
\theta_2 &= \frac{T_z y - T_y f}{Z} + \omega_x f - \omega_z x - \frac{\omega_y x y}{f} + \frac{\omega_x y^2}{f}
\end{aligned}$$

- Linear in motion terms
- Complicated due to depth dependence

# Motion Field and Optic Flow

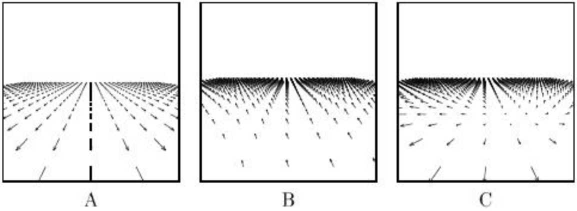



**Figure 2:** Example flow fields. **A:** Camera translation. **B:** Camera rotation. **C:** Translation plus rotation. Each flow vector in C is the vector sum of the two corresponding flow vectors in A and B.

Illustrative rigid motion flow fields

# Motion Field and Optic Flow

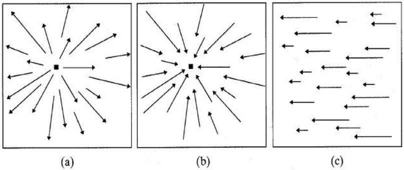



Figure 8.4 The three types of motion field generated by translational motion. The filled square marks the instantaneous epipole.

## Pure 3D Translation

No rotation, $\therefore$ flow is

$$
\begin{aligned}
\boldsymbol{\theta}(x,y) &= p(x,y)\boldsymbol{A}(x,y)\mathbf{T} + \cancel{\boldsymbol{B}(x,y)\boldsymbol{\Omega}}^{\,0} \\
\theta_1(x,y) &= p(x - x_0)T_z \\
\theta_2(x,y) &= p(y - y_0)T_z
\end{aligned}
$$

where $x_0 = \frac{fT_x}{T_z}$ and $y_0 = \frac{fT_y}{T_z}$

# Motion Field and Optic Flow

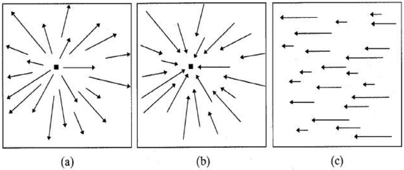



Figure 8.4 The three types of motion field generated by translational motion. The filled square marks the instantaneous epipole.

## Pure 3D Translation

$$\begin{aligned}\theta_1(x,y) &= p(x-x_0)T_z \\ \theta_2(x,y) &= p(y-y_0)T_z\end{aligned}$$

- Flow towards or away from *focus of expansion* (FOE)
- Defines the instantaneous epipole
- Magnitude of flow vector depends on ?

### Pure 3D Rotation

- Motion field independent of point depth
- Depends only on axis and angle of rotation
- Flow field is a quadratic function of image position

# Motion Field and Optic Flow

## Optic Flow

- optic flow is flow field in image
- Not the same as motion field
- optic flow is purely an image measure
- Flow field can occur due to variety of factors
  - Rigid camera motion
  - Rigid or non-rigid object motion (deformation)
  - Moving fluids (clouds, water etc.)
  - Articulated objects (moving hand, beating heart etc.)

# Motion Field and Optic Flow

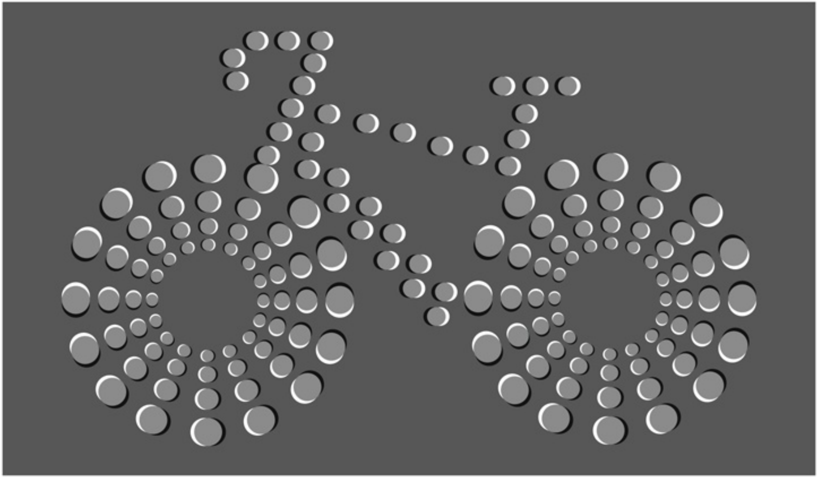


Image by S. Kitaoka from *Foundations of Computer Vision*

## Optic Flow

- Optic Flow is an image based measure
- How do we measure a purely image-based flow?
- Simple constraint obtained
- In general, ill-posed problem
- Can impose a range of constraints
- Leads to a variety of algorithms
- Tremendous amount of work on computational questions
- Issues of robustness etc.

### Modeling Image Flow

- Dense measurement of instantaneous image motion
- Flow field $\Rightarrow$ Correspondence across two images
- Flow field defined by $u(x, y)$ and $v(x, y)$

**Fundamental Assumption :** $I(x, y, t) = I(x + u, y + v, t + 1)$

- Conservation of image intensity
- Matched points have the same brightness or intensity

# Motion Field and Optic Flow

$$I(x, y, t) = I(x + u, y + v, t + 1)$$

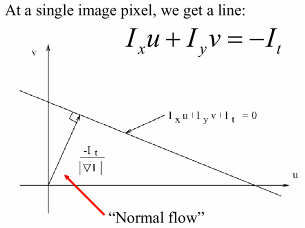
First-order approximation of the image function

$$I(x + u, y + v, t + 1) \approx I(x, y, t) + \frac{\partial I}{\partial x} u + \frac{\partial I}{\partial y} v + \frac{\partial I}{\partial t}.1$$
$$\Rightarrow \frac{\partial I}{\partial x} u + \frac{\partial I}{\partial y} v + \frac{\partial I}{\partial t}.1 = 0$$

**Brightness Constraint**

$$\frac{\partial I}{\partial x} u + \frac{\partial I}{\partial y} v + \frac{\partial I}{\partial t}.1 = 0$$

# Motion Field and Optic Flow




- Constraint $I_x u + I_y v + I_t = 0$
- Provides a single constraint (line)
- Flow at each point is defined by two variables

## Aperture Problem

- Flow estimation is *ill posed*
- Can measure accurately in one direction (normal flow)
- Perpendicular direction is completely ambiguous (aperture problem)

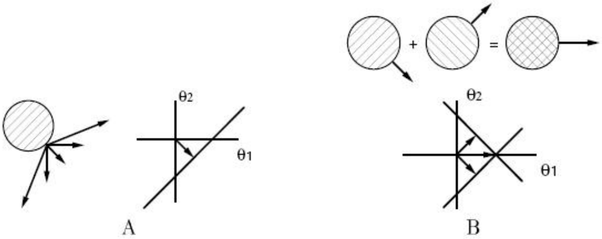



**Figure 5:** Aperture problem. **A:** Single moving grating viewed through a circular aperture. The diagonal line indicates the locus of velocities compatible with the motion of the grating. **B:** Plaid composed of two moving gratings. The lines give the possible motion of each grating alone. Their intersection is the only shared motion.

### Problem

Make flow estimation well-posed

### Estimation Approach

Impose assumptions on flow field

### Implicit assumptions of smoothness

**Lower order parametric representations**

- Constant Flow : Lucas-Kanade algorithm
- Affine Flow : Parametric Model

### Explicit assumptions of smoothness

**Variational approaches**

- Smooth Flow : Horn-Schunck algorithm

# Motion Field and Optic Flow

## Constant Flow : Lucas-Kanade Algorithm

- Simplest assumption imposed on flow
- Flow is considered constant within a patch (window W)
- Replace

$$I_x u(x,y) + I_y v(x,y) + I_t = 0$$

with

$$I_x{}^i u + I_y{}^i v + I_t{}^i = 0, \forall i \in W$$

Sum of square differences as error measure

$$
\begin{aligned}
E(u,v) &= \sum_{i \in W} ||I_1(x,y,t) - I_2(x+u, y+v, t+1)||^2 \\
&= \sum_{i \in W} (I_x u + I_y v + I_t)^2
\end{aligned}
$$

**Constant Flow : Lucas-Kanade Algorithm**

- Assume flow is constant within a patch
- Need to solve for 2 values of flow $(u, v)$
- Error measure can be thought of as sum-of-squares-difference
- Error measure can be written as “brightness constraint” deviation

# Motion Field and Optic Flow

Sum of square differences as error measure

$$
\begin{aligned}
E(u,v) &= \sum_{i \in W} ||I_1(x,y,t) - I_2(x+u, y+v, t+1)||^2 \\
&= \sum_{i \in W} (I_x u + I_y v + I_t)^2
\end{aligned}
$$

To solve, take derivatives wrt u and v

$$
\begin{aligned}
\frac{\partial E}{\partial u} &= \sum_{i \in W} 2I_x(I_x u + I_y v + I_t) = 0 \\
\frac{\partial E}{\partial v} &= \sum_{i \in W} 2I_y(I_x u + I_y v + I_t) = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E}{\partial u} &= \sum_{i \in W} 2I_x(I_x u + I_y v + I_t) = 0 \\
\frac{\partial E}{\partial v} &= \sum_{i \in W} 2I_y(I_x u + I_y v + I_t) = 0
\end{aligned}
$$

Can rewrite as

$$
\begin{bmatrix} \sum {I_x}^2 & \sum I_x I_y \\ \sum I_x I_y & \sum {I_y}^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} \sum I_x I_t \\ \sum I_y I_t \end{bmatrix}
$$

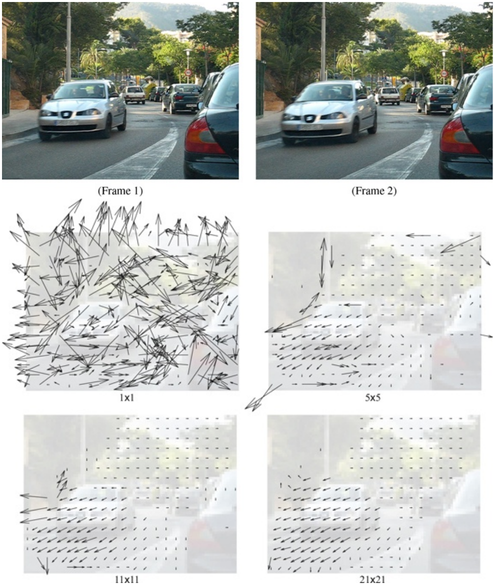
# Motion Field and Optic Flow

$$\begin{bmatrix} \sum {I_x}^2 & \sum I_x I_y \\ \sum I_x I_y & \sum {I_y}^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} \sum I_x I_t \\ \sum I_y I_t \end{bmatrix}$$

## Constant Flow : Lucas-Kanade algorithm

- Solve this linear system
- Matrix entries are averaged over patch
- Matrix is now full rank due to local averaging
- What does the matrix look like ?
- Finding "best" intersection of brightness constraints
- Each pixel in patch can be weighted using a mask

# Motion Field and Optic Flow




**What about patch size?**

$$\begin{bmatrix} u \\ v \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} x \\ y \end{bmatrix} + \boldsymbol{t}$$

### Affine Flow

- Model flow in patch with affine form (6 parameters)
- Much more flexible than constant flow model
- Flow here is linear in affine parameters
- Can solve by plugging into brightness constraint
- Use earlier error measure
- Can model complex flows more accurately
- Need more equations (larger patch) for better fit

# Motion Field and Optic Flow

## Affine Flow Model

$$\begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

## Restacking we have

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}}_{\boldsymbol{s}} = \underbrace{\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ 1 \end{bmatrix}}_{\boldsymbol{a}}$$

$$\Rightarrow \boldsymbol{s} = \boldsymbol{P}\boldsymbol{a}$$

## Affine Flow Model

$$\underbrace{\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}}_{\boldsymbol{s}} = \underbrace{\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ 1 \end{bmatrix}}_{\boldsymbol{a}}$$

- Brightness Constraint is $\begin{bmatrix} I_x & I_y & I_t \end{bmatrix} \boldsymbol{P}\boldsymbol{a} = 0$
- Minimise lsq error of form $\boldsymbol{a}^T \boldsymbol{M} \boldsymbol{a}$

# Motion Field and Optic Flow

$$E(u,v) = \int_{\Omega}((I_x u + I_y v + I_t)^2 + \alpha(|\nabla u|^2 + |\nabla v|^2))dxdy$$

## Smooth Flow Fields : Horn and Schunck

- Solves for smooth flow field
- Smoothness term takes care of *aperture problem*
- Cost function solved using *calculus of variations*
- Iterative solutions (using Euler-Lagrange equations or variations)
- $\alpha$ controls degree of smoothness

## Optic Flow Estimation Issues

- Correcting for intensity variations
- Filters for accurate derivative computation
- Use spatio-temporal filters
- Robustness
- Multi-scale refinement
- Efficient optimisation
- Very sophisticated numerical methods used